

4.4 One-sided limits

Definition 4.4.1. A function f has the limit $L \in \mathbb{R}$ as x approaches a real number a from the left if the following two conditions are satisfied:

- (I) There exists a real number $\delta_0 > 0$ such that $f(x)$ is defined for each x in the set $(a - \delta_0, a)$.
- (II) For each real number $\epsilon > 0$ there exists a real number $\delta(\epsilon)$ such that $0 < \delta(\epsilon) \leq \delta_0$ and

$$0 < a - x < \delta(\epsilon) \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

If the conditions (I) and (II) in Definition 4.4.1 are satisfied we write $\lim_{x \uparrow a} f(x) = L$.

Definition 4.4.2. A function f has the limit $L \in \mathbb{R}$ as x approaches a real number a from the right if the following two conditions are satisfied:

- (I) There exists a real number $\delta_0 > 0$ such that $f(x)$ is defined for each x in the set $(a, a + \delta_0)$.
- (II) For each real number $\epsilon > 0$ there exists a real number $\delta(\epsilon)$ such that $0 < \delta(\epsilon) \leq \delta_0$ and

$$0 < x - a < \delta(\epsilon) \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

If the conditions (I) and (II) in Definition 4.4.2 are satisfied we write $\lim_{x \downarrow a} f(x) = L$.

Definition 4.4.3. A function f has the limit $+\infty$ as x approaches a real number a from the left if the following two conditions are satisfied:

- (I) There exists a real number $\delta_0 > 0$ such that $f(x)$ is defined for each x in the set $(a - \delta_0, a)$.
- (II) For each real number $M > 0$ there exists a real number $\delta(M)$ such that $0 < \delta(M) \leq \delta_0$ and

$$0 < a - x < \delta(M) \quad \Rightarrow \quad f(x) > M.$$

If the conditions (I) and (II) in Definition 4.4.3 are satisfied we write $\lim_{x \uparrow a} f(x) = +\infty$.

Definition 4.4.4. A function f has the limit $+\infty$ as x approaches a real number a from the right if the following two conditions are satisfied:

- (I) There exists a real number $\delta_0 > 0$ such that $f(x)$ is defined for each x in the set $(a, a + \delta_0)$.
- (II) For each real number $M > 0$ there exists a real number $\delta(M)$ such that $0 < \delta(M) \leq \delta_0$ and

$$0 < x - a < \delta(M) \quad \Rightarrow \quad f(x) > M.$$

If the conditions (I) and (II) in Definition 4.4.4 are satisfied we write $\lim_{x \downarrow a} f(x) = +\infty$.

Definition 4.4.5. A function f has the limit $-\infty$ as x approaches a real number a from the left if the following two conditions are satisfied:

- (I) There exists a real number $\delta_0 > 0$ such that $f(x)$ is defined for each x in the set $(a - \delta_0, a)$.
- (II) For each real number $M < 0$ there exists a real number $\delta(M)$ such that $0 < \delta(M) \leq \delta_0$ and

$$0 < a - x < \delta(M) \quad \Rightarrow \quad f(x) < M.$$

If the conditions (I) and (II) in Definition 4.4.5 are satisfied we write $\lim_{x \uparrow a} f(x) = -\infty$.

Definition 4.4.6. A function f has the limit $-\infty$ as x approaches a real number a from the right if the following two conditions are satisfied:

- (I) There exists a real number $\delta_0 > 0$ such that $f(x)$ is defined for each x in the set $(a, a + \delta_0)$.
- (II) For each real number $M < 0$ there exists a real number $\delta(M)$ such that $0 < \delta(M) \leq \delta_0$ and

$$0 < x - a < \delta(M) \quad \Rightarrow \quad f(x) < M.$$

If the conditions (I) and (II) in Definition 4.4.6 are satisfied we write $\lim_{x \downarrow a} f(x) = -\infty$.

Exercise 4.4.7. Find each of the following limits. Prove your claims using the appropriate definition.

- (a) $\lim_{x \uparrow 5} \frac{3x - 15}{\sqrt{x^2 - 10x + 25}}$ (b) $\lim_{x \downarrow 5} \frac{3x - 15}{\sqrt{x^2 - 10x + 25}}$ (c) $\lim_{x \uparrow 2} \frac{x - 3}{x(x - 2)}$
- (d) $\lim_{x \downarrow 0} \left(\frac{1}{x} - \frac{1}{x^2} \right)$ (e) $\lim_{x \uparrow 5} \frac{2}{\sqrt{5 - x}}$ (f) $\lim_{x \downarrow 5} \frac{6}{5 - x}$
- (g) $\lim_{x \uparrow 3} \frac{x + 3}{x^2 - 9}$ (h) $\lim_{x \uparrow -3} \frac{x^2}{x^2 - 9}$ (i) $\lim_{x \downarrow 0} (x - \sqrt{x})$
- (j) $\lim_{x \rightarrow 3} \frac{x}{(x - 3)^2}$ (k) $\lim_{x \downarrow -1} \frac{x^2}{x + 1}$ (l) $\lim_{x \rightarrow +\infty} (x - \sqrt{x})$

5 Continuous Functions

5.1 Definition and Examples

All this work about limits will now pay off since we shall be able to give mathematically rigorous definition of a continuous function.

Definition 5.1.1. Let f be a real valued function of a real variable and let a be a real number. The function f is continuous at a if the following two conditions are satisfied:

- (i) The function f is defined at a , that is $f(a)$ is defined.
- (ii) $\lim_{x \rightarrow a} f(x) = f(a)$.

To understand Definition 5.1.1 the reader has to understand the concept of limit. Sometimes it is useful to state the definition of continuity directly, without appealing to the concept of limit.

Definition 5.1.2. Let f be a real valued function of a real variable and let a be a real number. The function f is continuous at a if the following two conditions are satisfied:

- (I) There exists a $\delta_0 > 0$ such that $f(x)$ is defined for all $x \in (a - \delta_0, a + \delta_0)$.
- (II) For each $\epsilon > 0$ there exists $\delta(\epsilon)$ such that $0 < \delta(\epsilon) \leq \delta_0$ and such that

$$|x - a| < \delta(\epsilon) \quad \Rightarrow \quad |f(x) - f(a)| < \epsilon.$$

Definition 5.1.2 is called ϵ - δ definition of continuity.

Example 5.1.3. Let c be a real number and define $f(x) = c$ for all $x \in \mathbb{R}$. Use Definition 5.1.2 to prove that f is continuous at an arbitrary real number a .

Example 5.1.4. Let $f(x) = x$ for all $x \in \mathbb{R}$. Use Definition 5.1.2 to prove that f is continuous at an arbitrary real number a .

Example 5.1.5. Use ϵ - δ definition of continuity, that is Definition 5.1.2, to prove that the function $f(x) = 1/x$ is continuous on the interval $(0, +\infty)$.

Solution. Let $a \in (0, +\infty)$, that is let a be an arbitrary positive number. Chose $\delta_0 = a/2$. Since $a > 0$, we conclude that $a/2 > 0$ and $f(x) = 1/x$ is defined for all $x \in (a/2, 3a/2)$.

Let $\epsilon > 0$ be arbitrary. Now we have to solve

$$\left| \frac{1}{x} - \frac{1}{a} \right| < \epsilon \quad \text{for} \quad |x - a|.$$

First simplify the expression, using the fact that $x > 0$ and $a > 0$ and rules for the absolute value:

$$\left| \frac{1}{x} - \frac{1}{a} \right| = \left| \frac{a - x}{x a} \right| = \frac{|a - x|}{|x| |a|} = \frac{|x - a|}{x a}.$$

To get a larger expression which will be easy to solve we replace x in the denominator by the smallest possible value for x . That value is $a - a/2 = a/2$. This gives me my BIN:

$$\left| \frac{1}{x} - \frac{1}{a} \right| = \frac{|x - a|}{x a} \leq \frac{|x - a|}{\frac{a}{2} a} = 2 \frac{|x - a|}{a^2}.$$

Thus my BIN is $\left| \frac{1}{x} - \frac{1}{a} \right| \leq 2 \frac{|x - a|}{a^2}$ valid for all $x \in (a/2, 3a/2)$.

Solving $2 \frac{|x - a|}{a^2} < \epsilon$ for $|x - a|$ is easy: The solution is $|x - a| < a^2 \epsilon/2$. Now we define

$$\delta(\epsilon) = \min \left\{ \frac{a^2 \epsilon}{2}, \frac{a}{2} \right\}.$$

It remains to prove the implication $|x - a| < \delta(\epsilon) \Rightarrow \left| \frac{1}{x} - \frac{1}{a} \right| < \epsilon$. This should be easy, using the BIN. \square

Example 5.1.6. Use ϵ - δ definition of continuity, that is Definition 5.1.2, to prove that the function $x \mapsto \sqrt{x}$ is continuous on the interval $(0, +\infty)$.

Solution. Let $a \in (0, +\infty)$. Chose $\delta_0 = \frac{a}{2}$. Since $a > 0$, as before we conclude that $\frac{a}{2} > 0$ and the function $x \mapsto \sqrt{x}$ is defined for all $x \in (a/2, 3a/2)$.

Let $\epsilon > 0$ be arbitrary. Now we have to solve

$$|\sqrt{x} - \sqrt{a}| < \epsilon \quad \text{for} \quad |x - a|.$$

First simplify algebraically the expression, using the fact that $x > 0$ and $a > 0$ and rules for the absolute value.

$$\begin{aligned} |\sqrt{x} - \sqrt{a}| &= \left| (\sqrt{x} - \sqrt{a}) \frac{1}{1} \right| = \left| (\sqrt{x} - \sqrt{a}) \frac{\sqrt{x} + \sqrt{a}}{\sqrt{x} + \sqrt{a}} \right| = \left| \frac{x - a}{\sqrt{x} + \sqrt{a}} \right| \\ &= \frac{|x - a|}{|\sqrt{x} + \sqrt{a}|} = \frac{|x - a|}{\sqrt{x} + \sqrt{a}} \leq \frac{|x - a|}{\sqrt{a}} \end{aligned}$$

Thus my BIN is: $|\sqrt{x} - \sqrt{a}| \leq \frac{|x - a|}{\sqrt{a}}$, valid for $x > 0$.

Solving $\frac{|x - a|}{\sqrt{a}} < \epsilon$ for $|x - a|$ is easy: The solution is $|x - a| < \sqrt{a} \epsilon$. Now we define

$$\delta(\epsilon) = \min \left\{ \sqrt{a} \epsilon, \frac{a}{2} \right\}.$$

It remains to prove the implication $|x - a| < \min \left\{ \sqrt{a} \epsilon, \frac{a}{2} \right\} \Rightarrow |\sqrt{x} - \sqrt{a}| < \epsilon$. This should be easy, using the BIN. \square

Example 5.1.7. Let $f(x) = \frac{1}{x^2 + 1}$ for all $x \in \mathbb{R}$. Use ϵ - δ definition to prove that f is continuous at an arbitrary $a \in \mathbb{R}$.

Theorem 5.1.8 (Algebra of Continuous Functions). *Let f and g be functions and let a be a real number. Assume that f and g are continuous at the point a .*

- (a) *If $h = f + g$, then h is continuous at a .*
- (b) *If $h = fg$, then h is continuous at a .*
- (c) *If $h = \frac{f}{g}$ and $g(a) \neq 0$, then h is continuous at a .*

Example 5.1.9. Let a, b, c be any real numbers. Let $f(x) = ax^2 + bx + c$ for all $x \in \mathbb{R}$. Let v be an arbitrary real number. Prove that f is continuous at v .

Example 5.1.10. Let $f(x) = \frac{1}{x^2 + 1}$ for all $x \in \mathbb{R}$. Use algebra of continuous functions to prove that f is continuous on \mathbb{R} .

Example 5.1.11. Let $f(x) = \sin x$ for all $x \in \mathbb{R}$. Prove that f is continuous at an arbitrary real number a .

Solution. The proof consists of two steps:

- (1) Use the definition of limit and inequalities that we proved for \sin and \cos to prove the following two limits

$$\begin{aligned}\lim_{x \rightarrow a} \sin(x - a) &= 0, \\ \lim_{x \rightarrow a} \cos(x - a) &= 1.\end{aligned}$$

- (2) Use the addition formula for $\sin x = \sin(x - a + a)$ and the algebra of limits to complete the proof. □

Example 5.1.12. Let $f(x) = \cos x$ for all $x \in \mathbb{R}$. Prove that f is continuous at an arbitrary real number a .

Example 5.1.13. Let $f(x) = \tan x$ for all $-\frac{\pi}{2} < x < \frac{\pi}{2}$. Prove that f is continuous at an arbitrary real number a such that $-\frac{\pi}{2} < a < \frac{\pi}{2}$.

Solution. Use the algebra of continuous functions. □

Example 5.1.14. Let $f(x) = \ln x$ for all $x \in (0, +\infty)$. Prove that f is continuous at an arbitrary real number a .

Solution. Use the definition of \ln to derive the squeeze for \ln :

$$1 - \frac{1}{x} \leq \ln x \leq x - 1, \quad 0 < x < +\infty.$$

Use the above squeeze to prove that for arbitrary $a > 0$ we have $\lim_{x \rightarrow a} \ln\left(\frac{x}{a}\right) = 0$. Now use the rule for logarithms $\ln uv = \ln u + \ln v$. □

Example 5.1.15. Let $f(x) = e^x$ for all $x \in \mathbb{R}$. Prove that f is continuous at an arbitrary real number a .

Solution. Use the fact that $x \mapsto e^x$ is the inverse of the logarithm function to derive the squeeze for it:

$$1 + x \leq e^x \leq \frac{1}{1 - x}, \quad -\infty < x < 1.$$

Get the rest of the proof as an exercise. □

Theorem 5.1.16. Let f and g be functions and let a be a real number. Assume that g is continuous at a and that f is continuous at $g(a)$. If $h = f \circ g$ then h is continuous at a .

6 New Limits from Old

6.1 Squeeze theorems

In this section and in Section 6.3 we establish general properties of limits which are based on Definition 4.1.1. These properties are stated as theorems.

Establishing theorems of this kind involves a major step forward in sophistication. Up to this point we have been trying to show that limits exist directly from the definition. Now for the first time we are going to **assume** that some limit exists (I refer to this in class as a *green* limit.) and try to make use of this information to establish the existence of some other limit (I refer to this in class as a *red* limit.). Remember that to establish the existence of a limit, we had to come up with a procedure for finding $\delta(\epsilon)$ that will work for any $\epsilon > 0$ that is given. If we assume the existence of a limit, then we are assuming the existence of such a procedure, though we may not know explicitly what it is. I refer to this as a *green* $\delta(\epsilon)$. It is this procedure we will need to use in order to construct a new procedure for the limit whose existence we are trying to establish. I refer to this as a *red* $\delta(\epsilon)$.

We start by considering squeeze theorems that resemble the role of BIN in previous sections. The following theorem is the Sandwich Squeeze Theorem.

Theorem 6.1.1. Let f, g and h be given functions and let a and L be real numbers. Suppose that the following three conditions are satisfied.

(1) $\lim_{x \rightarrow a} f(x) = L,$

(2) $\lim_{x \rightarrow a} h(x) = L,$

(3) There exists $\eta_0 > 0$ such that $f(x), g(x)$ and $h(x)$ are defined for all $x \in (a - \eta_0, a) \cup (a, a + \eta_0)$ and

$$f(x) \leq g(x) \leq h(x) \quad \text{for all } x \in (a - \eta_0, a) \cup (a, a + \eta_0).$$

Then

$$\lim_{x \rightarrow a} g(x) = L.$$

Proof. Here we have three functions and three definitions of limits, one for each function. Therefore we have to deal with three δ -s. We shall give them appropriate names that will distinguish them from each other. Let us name them δ_f, δ_g and δ_h .

In the theorem it is assumed that $\lim_{x \rightarrow a} f(x) = L$. This means that we are given the fact that for each $\epsilon > 0$ there exists $\delta_f(\epsilon) > 0$ (that is, we are given a function $\delta_f(\epsilon)$) such that

$$0 < |x - a| < \delta_f(\epsilon) \quad \Rightarrow \quad |f(x) - L| < \epsilon. \quad (6.1.1)$$

In class I refer to these as a green $\delta_f(\cdot)$ and a green implication.

Since the theorem assumes that $\lim_{x \rightarrow a} h(x) = L$, we are also given that for each $\epsilon > 0$ there exists $\delta_h(\epsilon) > 0$ such that

$$0 < |x - a| < \delta_h(\epsilon) \quad \Rightarrow \quad |h(x) - L| < \epsilon. \quad (6.1.2)$$

Again we refer to these as a green $\delta_h(\cdot)$ and a green implication.

We need to prove that $\lim_{x \rightarrow a} g(x) = L$. Therefore, following Definition 4.1.1, we have to show that the following conditions are satisfied:

(I) There exists a real number $\delta_{0,g} > 0$ such that $g(x)$ is defined for each x in the set $(a - \delta_{0,g}, a) \cup (a, a + \delta_{0,g})$.

(II) For each real number $\epsilon > 0$ there exists a real number $\delta_g(\epsilon)$ such that $0 < \delta_g(\epsilon) \leq \delta_{0,g}$ and such that

$$0 < |x - a| < \delta_g(\epsilon) \quad \Rightarrow \quad |g(x) - L| < \epsilon. \quad (6.1.3)$$

Since we have to produce $\delta_{0,g}, \delta_g(\epsilon)$ and we have to prove the last implication, all of these objects are red.

Notice that η_0 in the theorem is green.

The objective here is to use the green objects to produce the red objects. We shall do that next. We put:

(I) $\delta_{0,g} = \eta_0$. By the assumption of the theorem $g(x)$ is defined for each x in the set $(a - \eta_0, a) \cup (a, a + \eta_0)$.

(II) For each real number $\epsilon > 0$, put

$$\delta_g(\epsilon) = \min\{\delta_f(\epsilon), \delta_h(\epsilon), \eta_0\}.$$

This is a beautiful expression since the red object is expressed in terms of the green objects.

It remains to prove the red implication (6.1.3) using the green implications and the assumptions of the theorem.

To prove (6.1.3), assume that $0 < |x - a| < \delta_g(\epsilon)$. Then, clearly, $0 < |x - a| < \eta_0$. This is telling me that $x \neq a$ and that x is no further than η_0 from a . Consequently, $x \in (a - \eta_0, a) \cup (a, a + \eta_0)$. Therefore, by the assumption of the theorem

$$f(x) \leq g(x) \leq h(x). \quad (6.1.4)$$

Subtracting L from each term in this inequality, I conclude that

$$f(x) - L \leq g(x) - L \leq h(x) - L. \quad (6.1.5)$$

Using the property of the absolute value that $-|u| \leq u \leq |u|$ for each real number u , we conclude that

$$-|f(x) - L| \leq f(x) - L \leq g(x) - L \leq h(x) - L \leq |h(x) - L|. \quad (6.1.6)$$

From the assumption $0 < |x - a| < \delta_g(\epsilon)$, we conclude that $0 < |x - a| < \delta_f(\epsilon)$. By the green implication (6.1.1), this implies that $|f(x) - L| < \epsilon$ and therefore

$$-\epsilon < -|f(x) - L|. \quad (6.1.7)$$

From the assumption $0 < |x - a| < \delta_g(\epsilon)$, we conclude that $0 < |x - a| < \delta_h(\epsilon)$. By the green implication (6.1.2), this implies that

$$|h(x) - L| < \epsilon. \quad (6.1.8)$$

Putting together the inequalities (6.1.6), (6.1.7) and (6.1.8), we conclude that

$$-\epsilon < g(x) - L < \epsilon. \quad (6.1.9)$$

The inequalities in (6.1.9) are equivalent to

$$|g(x) - L| < \epsilon. \quad (6.1.10)$$

This proves that $0 < |x - a| < \delta_g(\epsilon)$ implies $|g(x) - L| < \epsilon$ and this is exactly the red implication (6.1.3). This completes the proof. \square

The following theorem is the Scissors Squeeze Theorem.

Theorem 6.1.2. *Let f, g and h be given functions and let $a \in \mathbb{R}$ and $L \in \mathbb{R}$. Assume that*

1. $\lim_{x \rightarrow a} f(x) = L$,
2. $\lim_{x \rightarrow a} h(x) = L$,
3. *There exists $\eta_0 > 0$ such that $f(x), g(x)$ and $h(x)$ are defined for all $x \in (a - \eta_0, a) \cup (a, a + \eta_0)$ and*

$$f(x) \leq g(x) \leq h(x) \quad \text{for all } x \in (a - \eta_0, a),$$

and

$$h(x) \leq g(x) \leq f(x) \quad \text{for all } x \in (a, a + \eta_0).$$

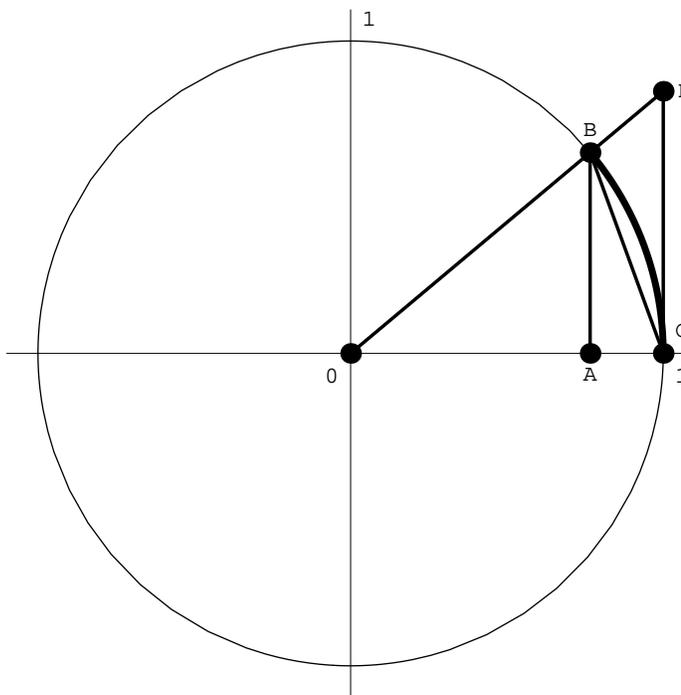
Then

$$\lim_{x \rightarrow a} g(x) = L.$$

6.2 Examples for Squeeze Theorems

The following picture and the numbers that *you can see on it* are essential for getting squeezes for limits involving trigonometric functions. The table to the left shows the numbers that you should be able to identify on the picture.

Geometric Object	Associated Number
Circular Arc from C to B	u
Line Segment \overline{OA}	$\cos u$
Line Segment \overline{AB}	$\sin u$
Line Segment \overline{AC}	$1 - \cos u$
Line Segment \overline{CB}	You Calculate
Line Segment \overline{CD}	$\tan u$
Line Segment \overline{OB}	1
Line Segment \overline{OC}	1



Example 6.2.1. Prove that $\lim_{x \rightarrow 0} \cos x = 1$.

Solution. Set $\eta_0 = \frac{\pi}{3}$. Consider positive u . Look at the picture above. The triangle $\triangle ACB$ is a right triangle. Therefore its hypotenuse, the line segment \overline{CB} , is longer than its side \overline{AC} which equals to $1 - \cos u$. Thus

$$1 - \cos u = \overline{AC} \leq \overline{CB}. \quad (6.2.1)$$

The line segment \overline{CB} is a segment of a straight line, therefore it is shorter than any other curve joining C and B . In particular it is shorter than the circular arc joining the points C and B . The length of this circular arc is u . Thus

$$\overline{CB} \leq \text{Length of the Circular Arc from } C \text{ to } B (= u). \quad (6.2.2)$$

Putting together the inequalities (6.2.1) and (6.2.2), we conclude that

$$1 - \cos u \leq u \quad \text{for all } 0 < u < \frac{\pi}{3}. \quad (6.2.3)$$

Since the length $\overline{OA} = \cos u$ is smaller than 1, from (6.2.3) we conclude that

$$0 \leq 1 - \cos u \leq u \quad \text{for all } 0 < u < \frac{\pi}{3},$$

or, equivalently,

$$1 - u \leq \cos u \leq 1 \quad \text{for all} \quad 0 < u < \frac{\pi}{3},$$

Now we substitute $u = |x|$ and use the fact that $\cos |x| = \cos x$ and (6.2) becomes

$$1 - |x| \leq \cos x \leq 1 \quad \text{for all} \quad -\frac{\pi}{3} < x < \frac{\pi}{3}. \quad (6.2.4)$$

This is a sandwich squeeze for $\cos x$. It is easy to prove that $\lim_{x \rightarrow 0} 1 = 1$ and $\lim_{x \rightarrow 0} (1 - |x|) = 1$. (Please prove this using the definition!) Now the Sandwich Squeeze Theorem implies that $\lim_{x \rightarrow 0} \cos x = 1$. \square

Example 6.2.2. Prove that $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$.

Solution. To get a sandwich squeeze for this problem consider the following three areas on the picture above.

Area 1 The triangle $\triangle OCB$.

Area 2 The segment of the unit disc bounded by the line segments \overline{OC} and \overline{OB} and the circular arc segment joining points C and B .

Area 3 The triangle $\triangle OCD$.

The picture tells clearly the inequality between these areas. Write that inequality. Calculate each area in terms of the numbers that appear in the table above. This will lead to the inequality, which when simplified gives

$$\cos u \leq \frac{\sin u}{u} \leq 1 \quad \text{for all} \quad 0 < u < \frac{\pi}{3}. \quad (6.2.5)$$

Using the same idea as in the previous example, the inequality (6.2.5) leads to

$$\cos x \leq \frac{\sin x}{x} \leq 1 \quad \text{for all} \quad x \in \left(-\frac{\pi}{3}, 0\right) \cup \left(0, \frac{\pi}{3}\right). \quad (6.2.6)$$

The inequality (6.2.6) is exactly what we need in the Sandwich Squeeze Theorem. Please fill in all the details of the rest of the proof. \square

Example 6.2.3. Prove that $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$.

Solution. To establish squeeze inequalities consider three lengths:

Length 1 The line segment \overline{AB} .

Length 2 The line segment \overline{CB} .

Length 3 The length of a circular arc joining the points C and B .

The picture tells clearly the inequalities between these three lengths. Write these inequalities. Calculate each length in terms of the numbers that appear in the table above. This will lead to the inequalities, which, when simplified, give

$$\frac{1}{2} \left(\frac{\sin u}{u} \right)^2 \leq \frac{1 - \cos u}{u^2} \leq \frac{1}{2} \quad \text{for all } 0 < u < \frac{\pi}{3}. \quad (6.2.7)$$

From the inequality (6.2.7) and one inequality established in a previous example you can get an “easy” sandwich squeeze. Please fill in all the details of the rest of the proof. \square

Example 6.2.4. Prove that $\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1$.

Solution. The idea is to use the definition of \ln as an integral and work with areas to get squeeze inequalities. \square

6.3 Algebra of limits

A nickname that I gave to a function which has a limit L when x approaches a is: f is *constantish* L near a . If we are dealing with constant functions $f(x) = L$ and $g(x) = K$, then clearly the sum $f + g$ of these two functions is a constant function equal to $L + K$. The same is true for the product fg which is the constant function equal to LK . Another question is whether we can talk about the reciprocal $1/f$. If $L \neq 0$, then the reciprocal of f is defined and it equals $1/L$. In this section we shall prove that all these properties hold for constantish functions.

Theorem 6.3.1. Let f, g , and h , be functions with domain and range in \mathbb{R} . Let a, K and L be real numbers. Assume that

$$(1) \quad \lim_{x \rightarrow a} f(x) = K,$$

$$(2) \quad \lim_{x \rightarrow a} g(x) = L.$$

Then the following statements hold.

$$(A) \quad \text{If } h = f + g, \text{ then } \lim_{x \rightarrow a} h(x) = K + L.$$

$$(B) \quad \text{If } h = fg, \text{ then } \lim_{x \rightarrow a} h(x) = KL.$$

$$(C) \quad \text{If } L \neq 0 \text{ and } h = \frac{f}{g}, \text{ then } \lim_{x \rightarrow a} h(x) = \frac{K}{L}.$$

Exercise 6.3.2. Use the algebra of limits to give much simpler proofs for most of the limits in the previous exercises and examples.